Journal of Statistical Physics, Vol. 59, Nos. 3/4, 1990

The Density of States in the Anderson Model at Weak Disorder: A Renormalization Group Analysis of the Hierarchical Model

Anton Bovier[1]

Received December 7, 1988; revision received December 1, 1989

We study the density of states in a hierarchical approximation of the Anderson tight-binding model at weak disorder using a renormalization group approach. Since the Laplacian term in our model is hierarchical, the renormalization group transformations act essentially on the local potential distribution and the energy. Technically, we use the supersymmetric replica trick and study the averaged Green's function. Starting with a Gaussian distribution with small variance, we find that the density of states is analytic as soon as the variance of the potential is turned on, except possibly near the band edge, where we can show this only for $\alpha > \sqrt{2}$, which corresponds to $d > 4$. Moreover, it is perturbatively close to the free one, except near the eigenvalues of the (hierarchical) Laplacian, where it is given (up to perturbative corrections) by the rescaled potential distribution.

KEY WORDS: Anderson model; density of states; weak disorder; hierarchical model; renormalization group.

1. INTRODUCTION

The Anderson model[1] is given by the random Hamiltonian (for reviews see, e.g., refs. 2)

$$H = -\Delta + V \tag{1.1}$$

on $l^2(\mathbb{Z}^d)$, where Δ is the finite-difference Laplacian and $V(x)$, $x \in \mathbb{Z}^d$, are independent, identically distributed random variables with a common dis-

[1] Physikalisches Institut, Universität Bonn, D-5300 Bonn, West Germany.

0022-4715/90/0500-0745$06.00/0

tribution μ. We will always choose μ with mean zero and denote its variance by $\sqrt{\lambda}$. In this paper we study a hierarchical version of this model,

$$H_\alpha = -\Delta_\alpha + V \tag{1.2}$$

on $l^2(\mathbb{Z})$, where Δ_α is the hierarchical Laplacian of Dyson,[3-5] and the potential is chosen as before. The precise definition of Δ_α will be given in Section 2. Note that although this model is defined on $\mathbb{Z}$, for different choices of the parameter α it is supposed to mimic the Anderson model in different dimensions (or, if preferred, a one-dimensional model with long-range hopping term). We also emphasize that our model is quite different from another type of hierarchical Anderson model, in which the random *potential* was chosen in a hierarchical way.[6,7]

In recent years there has been a great deal of progress in the mathematical understanding of the properties of the Anderson model. The results concern the question of "localization" as well as the smoothness properties of the density of states. Localization has been demonstrated in $d = 1$, or else for λ large or for energies well outside the band, for large classes of distributions μ.[8-11] For the density of states, again much is known about the smoothness properties in one dimension[10,12-16]; in higher dimension, results have required either λ or E large,[17-19] with the exception of the general bound of Wegner[20] (see also ref. 21 for a generalization), which shows that the density of states has a bounded derivative, provided the distribution μ is absolutely continuous with a bounded density.

Most of these results are obtained by considering $-\Delta$ as a perturbation of the potential term. The converse situation, where the potential should be a weak perturbation of the laplacian, has been very difficult to analyze. In one dimension, the perturbation expansion for the density of states has been investigated extensively,[22-25] but even there the results are not completely satisfactory. In particular, the question of whether for Bernoulli distributed potentials the density of states is smooth at weak disorder is still open. In higher dimensions, we are not aware of any rigorous results.

In particular, there exists no proof that extended states exist for sufficiently small disorder, in any dimension (there are results to that extent for the Bethe lattice[26,27]; however, a complete proof has not been published).

In this situation a better understanding of the weak-disorder regime of the Anderson model is extremely desirable. We believe that a viable approach to this problem must follow the ideas of the renormalization group. With the present paper we establish a first step toward this goal. We consider the presumably simpler problem of studying the Green's function (density of states) and we study the hierarchical approximation to the full model. The results we obtain in this case are very encouraging and confirm our expectations.

We would like at this point to give a general outline of our approach. The density of states $N(E)$ is defined as

$$N(E) = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \{ \# \text{eigenvalues of } H(\Lambda) \leqslant E \} \tag{1.3}$$

It is known that this quantity exists and is almost surely independent of the potential.[28-31] Moreover, the averaged Green's function

$$G(\zeta) = \int d\mu(V) \lim_{|\Lambda| \to \infty} \left(\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle x | (H - \zeta)^{-1} | x \rangle \right) \tag{1.4}$$

is the Borel transform of the measure $dN(E)$ (see, e.g., ref. 31), i.e.,

$$G(E) = \int \frac{dN(z)}{E - z} \tag{1.5}$$

and if $dN_{\text{a.c.}}$ and dN_{sing} denote the absolutely continuous and the singular part of dN, respectively, then

$$\frac{dN_{\text{a.c.}}(E)}{dE} = \frac{1}{\pi} \Im G(E + i0) \tag{1.6}$$

and dN_{sing} is supported by the set

$$\{ E \in \mathbb{R} \mid \lim_{\varepsilon \to 0} \Im G(E + i\varepsilon) = \infty \}$$

We will therefore always consider the averaged Green's function. To compute it, we write the Green's function in terms of a functional integral over "superfields." This "supersymmetric replica trick" goes back to Parisi and Sourlas,[32-34] and has been used frequently in this context.[15,16,19,35-37] It allows one to compute the average over the random fields. We take the distribution of the potential to be Gaussian, and thus end up with a supersymmetric Φ^4 model.

Arranging the functional integral over the superfields in a hierarchical fashion, we derive the renormalization group transformations. Since our Laplacian is chosen hierarchically, the renormalization group transformations preserve the structure of this model and only renormalize the local terms in it, i.e., the energy and the Fourier transform of the potential distribution. Thus, the map is just acting in a space of functions. As in the case of usual lattice field theories, one has now to control the action of this map. For an excellent review, which has largely inspired our procedure technically, we suggest the lecture notes by Gawedzki and Kupiainen[38]; see also refs. 4 and 5.

The hierarchical Laplacian has a discrete spectrum with one accumulation point, which corresponds to the (only) extended state for this operator. We refer to it also as the "band edge." Then there are three physical situations to be distinguished: (i) the energy is well away from any of the eigenvalues of the free Laplacian; (ii) the energy is close to or at one of the isolated eigenvalues; (iii) the energy is close to the accumulation point of the free spectrum.

In case (i) we expect nothing interesting; the Green's function should be perturbatively close to the free one. We show that this is indeed the case.

In case (ii) the situation is more complicated, since $(-\Delta_\alpha + E)$ is vanishingly small on some states. The best we can hope for is that the random potentials save the resolvent from diverging. The pole in the free Green's function should thus be replaced by a contribution that is roughly of size $1/\sqrt{\lambda}$. We show that this indeed happens. In fact we will see that perturbation theory breaks down exactly in the renormalization group step in which the hierarchy that is in resonance with the present energy is treated. Localizing the problem to this one step allows us to solve it by performing one "nonperturbative" step.

Case (iii) is the most difficult one to analyze. Intuitively, one sees that this region should be accessible to this method only if the nonperturbative regions around the eigenvalues stay well separated and do not coalesce. We will see that this is the case if $\alpha > \sqrt{2}$ ($\alpha > \sqrt{2}$ corresponds to $d > 4$ in the real model.). In the language of the renormalization group, this case can be distinguished also in the following way: In the two previous cases, the energy is the most relevant parameter and scales to infinity. All other parameters grow much more slowly (if at all), and are therefore "relatively irrelevant." Here, the energy is tuned to scale into a fixpoint, corresponding to the accumulation point. Therefore we have to consider the second most relevant parameter, which corresponds to the variance of μ. Our approach is immediately successful only if it is irrelevant, i.e., it scales to zero, which is the case for $\alpha > \sqrt{2}$. The marginal situation with $\alpha = \sqrt{2}$ could in principal be studied by computing higher orders in perturbation theory, but we have not done this.

Case (iii) is most interesting also because it concerns the extended state of Δ_α. A natural guess is that for $\alpha > \sqrt{2}$ it survives in the presence of a weak potential, whereas for $\alpha < \sqrt{2}$ it may become localized. The intuitive argument is that since the "spectral lines" in the Green's function remain distinct and associated with the particular hierarchy levels, so do the corresponding eigenstates, and a state corresponding to the "last" level exists for all sizes of the box. This would then be the extended state. It would be most interesting to prove such a result.

We summarize our results in the following theorem:

Theorem 1. Let μ be the Gaussian distribution with variance λ. Then:

(i) If $\alpha > \sqrt{2}$, there exists $\lambda_0 > 0$ such that, for $0 < \lambda < \lambda_0$, $G(\zeta)$ is an entire function.

(ii) For $\alpha \leqslant \sqrt{2}$, for any $\varepsilon > 0$ there exists λ_0 such that, for $0 < \lambda < \lambda_0$, $G(\zeta)$ is analytic for $|\zeta - E_\infty| > \varepsilon$, where E_∞ is the accumulation point in the spectrum of $-\Delta_\alpha$.

(iii) In the respective regions, $G^\alpha_\infty(\zeta)$ admits an asymptotic expansion in $\sqrt{\lambda}$ about a leading term that is singular in λ. This leading contribution gives for the differentiated density of states

$$\frac{dN(E)}{dE} \approx \frac{1}{(\pi\lambda)^{1/2}} \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^n} \exp\left[-\frac{2^n}{4\lambda}(E - E_{n-1})^2\right] \tag{1.7}$$

where E_n denotes the nth eigenvalue of $-\Delta_\alpha$.

The remainder of this paper is devoted to proving Theorem 1. In Section 2 we introduce the necessary concepts from supersymmetry, define the hierarchical Laplacian in this language, and compute its Green's function. Then we define the hierarchical Anderson model and derive the renormalization group equations. In Section 3 we compute perturbatively the renormalization group flow and hence the Green's function to leading orders. In Section 4 we explain why the approximations made in Section 3 are justified, i.e., we show how the nonperturbative contributions are controlled.

2. THE MODEL AND SOME FORMULAS

In this section we present the basic definitions for our model. Although all of this material can be found in the literature, we include it here for consistency and for the convenience of the reader. We begin by describing the hierarchical Laplacian in the supersymmetric formulation.

2.1. Some Superformulas

As usual, let Φ denote a superfield, i.e.,

$$\Phi = (\vec{\phi}, \psi, \bar{\psi}) \tag{2.1}$$

where $\vec{\phi}$ is a two-dimensional scalar field, and ψ, $\bar{\psi}$ is a pair of anticommuting variables, i.e., elements of a Grassmann algebra satisfying

$$\psi(x)\,\psi(y) = -\psi(y)\,\psi(x), \quad \text{etc.} \tag{2.2}$$

Functions of superfields are defined through their Taylor expansion in the anticommuting variables, e.g.,

$$F(\Phi(x)) = f(\vec{\phi}(x)) + \psi(x)\,h(\vec{\phi}(x)) + \bar{\psi}(x)\,k(\vec{\phi}(x)) + \bar{\psi}(x)\,\psi(x)\;g(\vec{\phi}(x)) \tag{2.3}$$

Defining the inner product

$$\Phi(x)\cdot\Phi(y) \equiv \vec{\phi}(x)\cdot\vec{\phi}(y) + \tfrac{1}{2}[\bar{\psi}(x)\,\psi(y) + \bar{\psi}(y)\,\psi(x)] \tag{2.4}$$

one can introduce the notion of supersymmetric functions: A function F is supersymmetric if it is a function of Φ^2 only, that is, if

$$F(\Phi) = f(\Phi^2) = f(\vec{\phi}^2) + \bar{\psi}\psi f'(\vec{\phi}^2) \tag{2.5}$$

There are two natural linear functionals defined for functions of supervariables, the ordinary Lebesgue integral w.r.t. the scalar field, and the Berezin integral[39] with respect to the anticommuting variables:

$$\int F(\Phi)\,d\bar{\psi}\,d\psi \equiv -g(\vec{\phi}) \tag{2.6}$$

For supersymmetric functions this reduces to[2]

$$\int f(\Phi^2)\,d^2\phi = \int f(\vec{\phi}^2)\,d^2\phi - 2\pi\bar{\psi}\psi f(0) \tag{2.7}$$

and

$$\int f(\Phi^2)\,d\bar{\psi}\,d\psi = -f'(\vec{\phi}^2) \tag{2.8}$$

so that the superintegral of a supersymmetric function yields simply

$$\int D\Phi\,f(\Phi^2) \equiv \int \frac{d\bar{\psi}\,d\psi}{\pi}\,d^2\phi\,f(\Phi^2) = f(0) \tag{2.9}$$

[2] We always assume that our functions vanish at infinity.

The main purpose of this formalism is that it allows us to write the following identity for the resolvent of a linear operator on $l^2(\Lambda)$:

$$\langle y| A^{-1} |z\rangle = i \int \prod_{x \in \Lambda} D\Phi(x)\, \psi(y)\, \bar{\psi}(z)\, e^{-i(\Phi, A\Phi)} \tag{2.10}$$

where

$$(\Phi, A\Phi) \equiv \sum_{x \in \Lambda} \Phi(x) \cdot A\Phi(x) \tag{2.11}$$

2.2. The Hierarchical Laplacian

We define now the hierarchical Laplacian of Dyson[3-5] in our formalism. While for the ordinary Laplacian restricted to $\Lambda \subset \mathbb{Z}^d$ we have

$$(\Phi, -\Delta\Phi) = -\sum_{\substack{\|x-y\|=1 \\ x, y \in \Lambda}} \Phi(x) \cdot \Phi(y) \tag{2.12}$$

the hierarchical Laplacian $-\Delta_\alpha$ has part of those terms removed and long-range interactions added. It is convenient to choose the box Λ to be $\Lambda_L \equiv [2^0, 2^L] \cap \mathbb{Z}$. Then

$$(\Phi, -\Delta_\alpha \Phi) = -\sum_{n=0}^{L-1} \sum_{l=1}^{2^{L-n-1}} \alpha^n \Phi_+^{(n)}(2l-1) \cdot \Phi_+^{(n)}(2l) \tag{2.13}$$

where the fields $\Phi_+^{(n)}$ are defined recursively by

$$\Phi_+^{(0)}(m) = \Phi(m)$$

$$\Phi_\pm^{(n)}(m) = \tfrac{1}{2}[\Phi_+^{(n-1)}(2m-1) \pm \Phi_+^{(n-1)}(2m)] \tag{2.14}$$

Note that

$$\Phi_+^{(n)}(2l-1) \cdot \Phi_+^{(n)}(2l) = [\Phi_+^{(n+1)}(l)]^2 - [\Phi_-^{(n+1)}(l)]^2 \tag{2.15}$$

and

$$\sum_{l=1}^{2^{L-n}} [\Phi_+^{(n)}(l)]^2 = 2 \sum_{l=1}^{2^{L-n-1}} [\Phi_+^{(n+1)}(l)]^2 + [\Phi_-^{(n+1)}(l)]^2 \tag{2.16}$$

Therefore

$$\begin{aligned}(\Phi, (-\Delta_\alpha - \zeta)\,\Phi) &= -\sum_{n=0}^{L-1} \sum_{l=1}^{2^{L-n-1}} \alpha^n \{[\Phi_+^{(n+1)}(l)]^2 - [\Phi_-^{(n+1)}(l)]^2\} \\ &\quad - \zeta \sum_{l=1}^{2^L} [\Phi^{(0)}(l)]^2 \\ &= -\alpha \sum_{n=1}^{L-1} \sum_{l=1}^{2^{L-n-1}} \alpha^{n-1} \{[\Phi_+^{(n+1)}(l)]^2 - [\Phi_-^{(n+1)}(l)]^2\} \\ &\quad - (2\zeta+1) \sum_{l=1}^{2^{L-1}} [\Phi_+^{(1)}(l)]^2 - (2\zeta-1) \sum_{l=1}^{2^{L-1}} [\Phi_-^{(1)}(l)]^2\end{aligned} \tag{2.17}$$

We define the spatially averaged Green's function[3]

$$G_L^\alpha(\zeta)=\frac{i}{2^L}\sum_{l=1}^{2^L}\int\prod_{m=1}^{2^L}D\Phi(m)\,\psi(l)\,\bar{\psi}(l)\,e^{-i(\Phi,(-\Delta_\alpha-\zeta)\Phi)} \tag{2.18}$$

As a consequence of (2.17), it satisfies

$$\begin{aligned}G_L^\alpha(\zeta)=i\int\prod_{m=1}^{2^{L-1}}&D\Phi_+^{(1)}(m)\,D\Phi_-^{(1)}(m)\\&\times\exp\left(i\sum_{n=1}^{L-1}\sum_{l=1}^{2^{L-n-1}}\alpha^n\{[\Phi_+^{(n+1)}(l)]^2-[\Phi_-^{(n+1)}(l)]^2\}\right)\\&\times\exp\left\{i\frac{(2\zeta+1)}{\alpha}\alpha\sum_{l=1}^{2^{L-1}}[\Phi_+^{(1)}(l)]^2+i(2\zeta-1)\sum_{l=1}^{2^{L-1}}[\Phi_-^{(1)}(l)]^2\right\}\\&\times\frac{1}{2^{L-1}}\sum_{l=1}^{2^{L-1}}\{\psi_+^{(1)}(l)\,\bar{\psi}_+^{(1)}(l)+\psi_-^{(1)}(l)\,\bar{\psi}_-^{(1)}(l)\}\end{aligned} \tag{2.19}$$

The integral over $\Phi_-^{(1)}$ can now be performed. This just means putting $\Phi_-^{(1)}(l)=0$ except in the term where $\psi_-^{(1)}(l)\,\bar{\psi}_-^{(1)}(l)$ appears. In the terms with no $\psi_+^{(1)}(l)\,\bar{\psi}_+^{(1)}(l)$ present, the $\Phi_+^{(1)}$ integrals can then also be performed, since their integrands are supersymmetric. The remaining terms are seen to reconstitute a Green's function on Λ_{L-1}, with ζ replaced by $(2\zeta+1)/\alpha$. We thus get

$$G_L^\alpha(\zeta)=\frac{1}{\alpha}G_{L-1}^\alpha\left(\frac{2\zeta+1}{\alpha}\right)+\frac{1}{1-2\zeta} \tag{2.20}$$

provided $L\geqslant 1$. For $L=0$, there is no Laplacian term and we get simply

$$G_0^\alpha(\zeta)=i\int D\Phi(1)\,\psi(1)\,\bar{\psi}(1)\,e^{i\zeta\Phi(1)^2}=-\frac{1}{\zeta} \tag{2.20a}$$

Defining ζ_n through

$$\begin{aligned}&\zeta_0=\zeta\\&\zeta_n=\frac{2\zeta_{n-1}+1}{\alpha}\qquad\text{for}\quad n\geqslant 1\end{aligned} \tag{2.21}$$

the recursion (2.20) has the explicit solution

$$G_L^\alpha(\zeta)=\sum_{n=0}^{L-1}\frac{1}{\alpha^n}\frac{1}{1-2\zeta_n}-\frac{1}{\alpha^L}\frac{1}{\zeta_L} \tag{2.22}$$

[3] We put $\zeta=E+i\varepsilon$, i.e., ζ is always supposed to have a positive imaginary part. We take ε to be zero at the end.

Thus, G_L^α has a singularity whenever $2\zeta_n = 1$ for some $0 \leqslant n \leqslant L-1$, or $\zeta_L = 0$. With the explicit solution

$$\zeta_n = \left(\frac{2}{\alpha}\right)^n \left(\zeta + \frac{1}{2-\alpha}\right) - \frac{1}{2-\alpha} \tag{2.23}$$

the singularities of the Green's function occur at the eigenvalues of $-\Delta_\alpha$, which we find thus to be

$$\begin{aligned} E_n &= \left(\frac{\alpha}{2}\right)^n \frac{2-\alpha/2}{2-\alpha} - \frac{1}{2-\alpha} \qquad \text{for} \quad 0 \leqslant n \leqslant L-1 \\ E_L &= \left(\frac{\alpha}{2}\right)^L \frac{1}{2-\alpha} - \frac{1}{2-\alpha} \end{aligned} \tag{2.24}$$

Thus, for $0 < \alpha < 2$, the spectrum of $-\Delta_\alpha$ is contained in the interval

$$\left[-\frac{1}{2-\alpha}, \frac{1-\alpha/2}{2-\alpha}\right]$$

and in the limit $L \to \infty$, $1/(\alpha - 2)$ is an accumulation point of the spectrum.

Combining (2.22) with (2.24) yields the following suggestive formula for the Green's function:

$$G_\infty^\alpha(\zeta) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \frac{1}{\zeta - E_n} \tag{2.25}$$

and as $\Im\zeta \to 0$, $\Re\zeta = E$,

$$G_\infty^\alpha(E + i0) = i\pi \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \delta(E - E_n) + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} P \frac{1}{E - E_n} \tag{2.26}$$

where P denotes the principal value. The (differentiated) density of states (i.e., the imaginary part of G) is thus a sum of delta functions with respective weights $1/2^{n+1}$, as expected.

2.3. The Hierarchical Anderson Model

The hierarchical version of the Anderson model we are considering is obtained by adding a random potential $V(l)$ to $-\Delta_\alpha$,

$$H_\alpha = -\Delta_\alpha + V \tag{2.27}$$

Here we take the $V(l)$ to be independent, identically distributed random variables, and their common distribution will be chosen to be a Gaussian with variance $\sqrt{\lambda}$.

We will compute the averaged Green's function [again denoted by $G_L^\alpha(\zeta)$]:

$$G_L^\alpha(\zeta)=\int d\mu(V)\, G_L^\alpha(\zeta;V)$$
$$=i\int\prod_{l=1}^{2^L} D\Phi(l)\, h(\Phi^2(l))\, e^{-i(\Phi,(-\Delta_\alpha-\zeta)\Phi)}\frac{1}{2^L}\sum_{m=1}^{2^L}\psi(m)\,\bar{\psi}(m) \tag{2.28}$$

where h denotes the characteristic function of the distribution μ. We let for the moment

$$h(\Phi^2(l))=e^{-\lambda\Phi^4(l)} \tag{2.29}$$

We can again try to organize the integrations over Φ in a hierarchical fashion. This yields

$$\begin{aligned}
G_L^\alpha(\zeta)=i\int\prod_{l=1}^{2^{L-1}} & D\Phi_+^{(1)}(l)\, D\Phi_-^{(1)}(l)\, h([\Phi_+^{(1)}(l)+\Phi_-^{(1)}(l)]^2)\\
&\times h([\Phi_+^{(1)}(l)-\Phi_-^{(1)}(l)]^2)\\
&\times\frac{1}{2^{L-1}}\sum_{m=1}^{2^{L-1}}[\psi_+^{(1)}(m)\,\bar{\psi}_+^{(1)}(m)+\psi_-^{(1)}(m)\,\bar{\psi}_-^{(1)}(m)]\\
&\times\exp\left\{i\sum_{n=1}^{L-1}\sum_{l=1}^{2^{L-n-1}}\alpha\alpha^{n-1}([\Phi_+^{(n+1)}(l)]^2-[\Phi_-^{(n+1)}]^2\right\}\\
&\times\exp\left\{i\frac{2\zeta+1}{\alpha}\alpha\sum_{l=1}^{2^{L-1}}[\Phi_+^{(1)}(l)]^2+i(2\zeta-1)\sum_{l=1}^{2^{L-1}}[\Phi_-^{(1)}(l)]^2\right\}\\
=i\int\prod_{l=1}^{2^{L-1}} & D\Phi_+^{(0)}(l)\, D\Phi_-^{(1)}(l)\, h\left(\frac{[\Phi_+^{(0)}(l)+\Phi_-^{(1)}(l)]^2}{\alpha}\right)\\
&\times h\left(\frac{[\Phi_+^{(0)}(l)-\Phi_-^{(1)}(l)]^2}{\alpha}\right)\\
&\times\exp\left\{i\sum_{n=0}^{L-2}\sum_{l=1}^{2^{L-n-2}}\alpha^n([\Phi_+^{(n+1)}(l)]^2-[\Phi_-^{(n+1)}(l)]^2)\right\}\\
&\times\exp\left\{i\frac{2\zeta+1}{\alpha}\sum_{l=1}^{2^{L-1}}[\Phi_+^{(0)}(l)]^2\right\}\\
&\times\frac{1}{2^{L-1}}\sum_{m=1}^{2^{L-1}}\left\{\frac{\psi_+^{(0)}(m)\,\bar{\psi}_+^{(0)}(m)}{\alpha}+\frac{\psi_-^{(1)}(m)\,\bar{\psi}_-^{(1)}(m)}{\alpha}\right\}\\
&\times\exp\left\{i\frac{2\zeta-1}{\alpha}\sum_{l=1}^{2^{L-1}}[\Phi_-^{(1)}(l)]^2\right\}
\end{aligned} \tag{2.30}$$

Here we assumed $L \geqslant 1$. For $L=0$, we get simply

$$G_0^\alpha(\zeta) = i \int D\Phi\, h(\Phi^2)\, e^{i\zeta\Phi^2} \psi\bar{\psi} \tag{2.30a}$$

We define

$$h^{(1)}(\Phi^2) = \int D\Phi_-\, e^{i[(2\zeta-1)/\alpha]\Phi_-^2} h\left(\frac{(\Phi+\Phi_-)^2}{\alpha}\right) h\left(\frac{(\Phi-\Phi_-)^2}{\alpha}\right) \tag{2.31}$$

and

$$g^{(1)}(\Phi) = \int D\Phi_-\, \psi_-\bar{\psi}_-\, e^{i[(2\zeta-1)/\alpha]\Phi_-^2} h\left(\frac{(\Phi+\Phi_-)^2}{\alpha}\right) h\left(\frac{(\Phi-\Phi_-)^2}{\alpha}\right) \tag{2.32}$$

so that

$$\begin{aligned}G_L^\alpha(\zeta) = {} & i \int \prod_{l=1}^{2^{L-1}} D\Phi(l) \exp\left\{i \sum_{n=0}^{L-2} \sum_{l=1}^{2^{L-n-2}} \alpha^n([\Phi_+^{(n+1)}(l)]^2 - [\Phi_-^{(n+1)}(l)]^2)\right\} \\ & \times \exp\left\{i \frac{2\zeta+1}{\alpha} \sum_{l=1}^{2^{L-1}} [\Phi(l)]^2\right\} \\ & \times \frac{1}{2^{L-1}} \sum_{m=1}^{2^{L-1}} \left\{\prod_{l=1}^{2^{L-1}} h^{(1)}([\Phi(l)]^2) \frac{\psi(m)\,\bar{\psi}(m)}{\alpha}\right. \\ & \left. + \prod_{l\neq m} h^{(1)}([\Phi(l)]^2) \frac{g^{(1)}([\Phi(m)])}{\alpha}\right\}\end{aligned} \tag{2.33}$$

If $g^{(1)}$ were supersymmetric, we could carry out the Φ integrations in the terms containing it and would just get $g^{(1)}(0)$. Our strategy will be to complete $g^{(1)}$ to a supersymmetric function by adding an appropriate pseudoscalar part and subtracting it from the terms containing $\bar{\psi}(m)\,\psi(m)$. In fact,

$$\begin{aligned}g^{(1)}(\Phi) &= \int \frac{d^2\phi_-}{\pi} \left[\exp\left(i \frac{2\zeta-1}{\alpha} \vec{\phi}_-^2\right)\right] h\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2 + \bar{\psi}\psi}{\alpha}\right) \\ & \quad \times h\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2 + \bar{\psi}\psi}{\alpha}\right) \\ &= \int \frac{d^2\phi_-}{\pi} \left[\exp\left(i \frac{2\zeta-1}{\alpha} \vec{\phi}_-^2\right)\right] h\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)\end{aligned}$$

$$\times h\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)-\frac{\psi\bar{\psi}}{\alpha}\int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta-1}{\alpha}\vec{\phi}_-^2\right)\right]$$
$$\times\left\{h'\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right.$$
$$\left.+h\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h'\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right\} \tag{2.34}$$

The derivative of the scalar part of this function is given by

$$\frac{1}{\alpha}\int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta-1}{\alpha}\vec{\phi}_-^2\right)\right]\left\{h'\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)\right.$$
$$\times h\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)+h\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h'\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)$$
$$-\frac{\vec{\phi}\cdot\vec{\phi}_-}{\phi^2}\left[h'\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right.$$
$$\left.\left.-h\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h'\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right]\right\} \tag{2.35}$$

Therefore we may write

$$g^{(1)}(\Phi)=g_0^{(1)}(\Phi^2)-\frac{\psi\bar{\psi}}{\alpha}\int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta-1}{\alpha}\vec{\phi}_-^2\right)\right]\frac{\vec{\phi}\cdot\vec{\phi}_-}{\vec{\phi}^2}$$
$$\times\left[h'\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right.$$
$$\left.-h\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h'\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right]$$
$$\equiv g_0^{(1)}(\Phi^2)-\frac{\psi\bar{\psi}}{\alpha}d^{(1)}(\phi^2) \tag{2.36}$$

where $g_0^{(1)}$ is the supersymmetric function with the same scalar part as $g^{(1)}$. Substituting this expression in (2.33), the integrals over the terms containing $g_0^{(1)}$ can now be performed. This gives, for $L\geqslant 1$,

$$G_L^\alpha(\zeta)=i\int\prod_{l=1}^{2^{L-1}}D\Phi(l)\exp\left\{i\sum_{n=0}^{L-2}\sum_{l=1}^{2^{L-n-2}}\alpha^n([\Phi_+^{(n+1)}(l)]^2-[\Phi_-^{(n+1)}(l)]^2\right\}$$
$$\times\exp\left\{i\frac{2\zeta+1}{\alpha}\sum_{l=1}^{2^{L-1}}[\Phi(l)]^2\right\}$$
$$\times\frac{1}{2^{L-1}}\sum_{m=1}^{2^{L-1}}\prod_{l\neq m}h^{(1)}([\Phi^2(l)])\frac{\psi(m)\,\bar{\psi}(m)}{\alpha}[h^{(1)}(\Phi^2(m))-d^{(1)}(\phi^2(m)]$$
$$+\frac{i}{\alpha}g^{(1)}(0) \tag{2.37}$$

The first term can be thought of as a modified Green's function

$$G^{\alpha}_{L-1}\left(\frac{2\zeta-1}{\alpha}; h^{(1)}, d^{(1)}\right)$$

on Λ_{L-1}. To obtain the general recursion formulas, we have to repeat the calculations leading to (3.37) taking the presence of the $d(\phi^2)$ term into account. Doing this, we find that, for $L-n+1 \geqslant 1$,

$$G^{\alpha}_{L-n+1}(\zeta_{n-1}; h^{(n-1)}, d^{(n-1)}) = \frac{1}{\alpha} G^{\alpha}_{L-n}(\zeta_n; h^{(n)}, d^{(n)}) + \frac{i}{\alpha}[g^{(n)}(0) - \delta g^{(n)}(0)] \qquad (2.38)$$

where

$$\zeta_n = \frac{2\zeta_{n-1}+1}{\alpha} \qquad (2.39)$$

$$h^{(n)}(\Phi^2) = \int D\Phi_- \exp\left[\left(i\frac{2\zeta_{n-1}-1}{\alpha}\Phi_-^2\right)\right] h^{(n-1)}\left(\frac{(\Phi+\Phi_-)^2}{\alpha}\right) \times h^{(n-1)}\left(\frac{(\Phi-\Phi_-)^2}{\alpha}\right) \qquad (2.40)$$

$$\begin{aligned} d^{(n)}(\phi^2) = &\int \frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta_{n-1}-1}{\alpha}\vec{\phi}_-^2\right)\right]\frac{\vec{\phi}\cdot\vec{\phi}_-}{\alpha\phi^2} \\ &\times\left[h^{(n-1)'}\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h^{(n-1)}\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right. \\ &\left. - h^{(n-1)}\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h^{(n-1)'}\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right] \\ &+ d^{(n-1)}\left(\frac{\phi^2}{\alpha}\right)h^{(n-1)}\left(\frac{\phi_-^2}{\alpha}\right) \\ &- \int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta_{n-1}-1}{\alpha}\vec{\phi}_-^2\right)\right]\left\{\frac{1}{2\alpha}\left[\frac{\vec{\phi}\cdot\vec{\phi}_-}{\phi^2}+\frac{\vec{\phi}\cdot\vec{\phi}_-}{\vec{\phi}_-^2}\right]\right. \\ &\times[d^{(n-1)}(+)\,h^{(n-1)'}(-) - d^{(n-1)'}(+)\,h^{(n-1)}(-) \\ &\left. - d^{(n-1)}(-)\,h^{(n-1)'}(+) + d^{(n-1)'}(-)\,h^{(n-1)}(+)]\right\} \end{aligned} \qquad (2.41)$$

$$g^{(n)}(0) = \int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta_{n-1}-1}{\alpha}\vec{\phi}_-^2\right)\right]\left[h^{(n-1)}\left(\frac{\phi_-^2}{\alpha}\right)\right]^2 \qquad (2.42)$$

and

$$\delta g^{(n)}(0) = \int \frac{d^2\phi_-}{\pi} \left[\exp\left(i \frac{2\zeta_{n-1} - 1}{\alpha} \vec{\phi}_-^2 \right) \right] h^{(n-1)}\left(\frac{\phi_-^2}{\alpha} \right) d^{(n-1)}\left(\frac{\phi_-^2}{\alpha} \right)$$

[We use the abbreviations $(+)$ and $(-)$ as arguments for $(\vec{\phi} \pm \vec{\phi}_-)^2/\alpha$ occasionally to keep our formulas more readable.]

For $n = L + 1$, i.e., the last step in the procedure, we get

$$\begin{aligned} G_0^\alpha(\zeta_L; h^{(L)}, d^{(L)}) &= i \int D\Phi\, e^{i\zeta_L \Phi^2} [h^{(L)}(\Phi^2) - d^{(L)}(\Phi^2)]\, \psi\bar{\psi} \\ &= -i \int \frac{d\phi^2}{\pi} e^{i\zeta_L \phi^2} [h^{(L)}(\phi^2) - d^{(L)}(\phi^2)] \\ &\equiv i[g^{(L+1)}(0) - \delta g^{(L+1)}(0)] \end{aligned} \tag{2.44}$$

where the definition of $g^{(L+1)}$ and $\delta g^{(L+1)}$ in the last line is of course a slight abuse of notation. It allows us, however, to express now the Green's function through the elegant formula

$$G_L^\alpha(\zeta) = i \sum_{n=1}^{L+1} \frac{1}{\alpha^n} [g^{(n)}(0) - \delta g^{(n)}(0)] \tag{2.45}$$

with $h^{(0)} = h$ and $d^{(0)} = 0$. This sets up our basic formalism. We see that we have an explicit formula for the Green's function in terms of the solution of the recursive functional equations (2.29)–(2.44). We study those equations in the next section.

3. THE PERTURBATIVE RENORMALIZATION GROUP FLOW

In this section we compute perturbatively $h^{(n)}(\Phi^2)$, $d^{(n)}(\phi^2)$, and hence $g^{(n)}(0)$ and $\delta g^{(n)}(0)$, starting with $h(\Phi^2) = e^{-\lambda \Phi^4}$ and expanding in small λ. All nonperturbative corrections will be ignored at this point; the control of the errors so produced is postponed to Section 4.

As a first step, we observe that $h^{(n)}$ are supersymmetric functions, and that we thus need only to compute their scalar parts. Carrying out the integrations over the fermionic variables, this gives

$$\begin{aligned} h^{(n)}(\vec{\phi}^2) = \int \frac{d^2\phi_-}{\pi} &\left[\exp\left(i \frac{2\zeta_{n-1} - 1}{\alpha} \vec{\phi}_-^2 \right) \right] \Bigg\{ -i \frac{2\zeta_{n-1} - 1}{\alpha} \\ &\times h^{(n-1)}\left(\frac{(\vec{\phi} + \vec{\phi}_-)^2}{\alpha} \right) h^{(n-1)}\left(\frac{(\vec{\phi} - \vec{\phi}_-)^2}{\alpha} \right) \\ &- \frac{1}{\alpha} \left[h^{(n-1)\prime}\left(\frac{(\vec{\phi} + \vec{\phi}_-)^2}{\alpha} \right) h^{(n-1)}\left(\frac{(\vec{\phi} - \vec{\phi}_-)^2}{\alpha} \right) \right. \\ &\left. + h^{(n-1)}\left(\frac{(\vec{\phi} + \vec{\phi}_-)^2}{\alpha} \right) h^{(n-1)\prime}\left(\frac{(\vec{\phi} - \vec{\phi}_-)^2}{\alpha} \right) \right] \Bigg\} \end{aligned} \tag{3.1}$$

or, after some partial integrations, in more suggestive form

$$h^{(n)}(\vec{\phi}^2)=\left[h^{(n-1)}\left(\frac{\vec{\phi}^2}{\alpha}\right)\right]^2+\int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta_{n-1}-1}{\alpha}\vec{\phi}_-^2\right)\right]$$
$$\times\frac{1}{\alpha}\frac{\vec{\phi}\cdot\vec{\phi}_-}{|\vec{\phi}_-|^2}\left[h^{(n-1)\prime}\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h^{(n-1)}\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right.$$
$$\left.-h^{(n-1)}\left(\frac{(\vec{\phi}+\vec{\phi}_-)^2}{\alpha}\right)h^{(n-1)\prime}\left(\frac{(\vec{\phi}-\vec{\phi}_-)^2}{\alpha}\right)\right] \qquad (3.2)$$

and

$$g^{(n)}(0)=\int\frac{d^2\phi_-}{\pi}\left[\exp\left(i\frac{2\zeta_{n-1}-1}{\alpha}\vec{\phi}_-^2\right)\right]\left[h^{(n-1)}\left(\frac{\vec{\phi}_-^2}{\alpha}\right)\right]^2 \qquad (3.3)$$

It seems useful to extract the "dominant" part of $h^{(n)}$ by defining

$$h^{(n)}(\vec{\phi}^2)=[\exp(-\lambda^{(n)}\vec{\phi}^4)]\,I^{(n)}(\vec{\phi}^2) \qquad (3.4)$$

where

$$\lambda^{(n)}=\left(\frac{2}{\alpha^2}\right)^n\lambda \qquad (3.5)$$

It is furthermore useful to perform some rescalings. First we introduce the functions $f^{(n)}$ through

$$I^{(n)}(\vec{\phi}^2)\equiv f^{(n)}((\lambda^{(n)})^{1/2}\,\vec{\phi}^2)\equiv f^{(n)}(s) \qquad (3.6)$$

We set

$$\frac{2\zeta_{n-1}-1}{\alpha(\lambda^{(n)})^{1/2}}\equiv z_n \qquad (3.7)$$

and change the integration variables to

$$t=(\lambda^{(n)})^{1/2}\,\vec{\phi}_-^2 \qquad (3.8)$$

and γ, the angle between $\vec{\phi}$ and $\vec{\phi}_-$. Our recursion, expressed for the functions $f^{(n)}$, takes the form

$$f^{(n)}(s)=\left[f^{(n-1)}\left(\frac{s}{\sqrt{2}}\right)\right]^2$$
$$-\frac{1}{2\pi}\int_0^{2\pi}d\gamma\int_0^{\infty}dt\{\exp[iz_nt-t^2-2st(1+2\cos^2\gamma)]\}\,4s\cos^2\gamma$$

$$\times f^{(n-1)}\left(\frac{s+t+2\cos\gamma(st)^{1/2}}{\sqrt{2}}\right) f^{(n-1)}\left(\frac{s+t-2\cos\gamma(st)^{1/2}}{\sqrt{2}}\right)$$

$$+\frac{1}{2\pi}\int_0^{2\pi} d\gamma \int_0^\infty dt\{\exp[iz_n t - t^2 - 2st(1+2\cos^2\gamma)]\}\left(\frac{s}{t}\right)^{1/2}\cos\gamma$$

$$\times\left\{f^{(n-1)'}\left(\frac{s+t+2\cos\gamma(st)^{1/2}}{\sqrt{2}}\right) f^{(n-1)}\left(\frac{s+t-2\cos\gamma(st)^{1/2}}{\sqrt{2}}\right)\right.$$

$$\left.-f^{(n-1)}\left(\frac{s+t+2\cos\gamma(st)^{1/2}}{\sqrt{2}}\right) f^{(n-1)'}\left(\frac{s+t-2\cos\gamma(st)^{1/2}}{\sqrt{2}}\right)\right\} \tag{3.9}$$

and

$$g^{(n)}(0) = -\frac{1}{(\lambda^{(n)})^{1/2}}\int_0^\infty dt\, e^{iz_n t - t^2}\left[f^{(n-1)}\left(\frac{t}{\sqrt{2}}\right)\right]^2 \tag{3.10}$$

In the same manner we may rewrite the recursion for $d^{(n)}$. We let

$$d^{(n)}(\phi^2) \equiv e^{-s^2}\tau^{(n)}(s) \tag{3.11}$$

The new functions $\tau^{(n)}$ then satisfy

$$\tau^{(n)}(s) = -\frac{1}{2\pi}\int_0^{2\pi} d\gamma \int_0^\infty dt\{\exp[iz_n t - t^2 - 2st(1+2\cos^2\gamma)]\}$$

$$\times\left\{4t\cos^2\gamma f^{(n-1)}(+)\, f^{(n-1)}(-)\right.$$

$$\left.-\left(\frac{t}{s}\right)^{1/2}\cos\gamma[f^{(n-1)'}(+)\, f^{(n-1)}(-) - f^{(n-1)}(+)\, f^{(n-1)'}(-)]\right\}$$

$$+\tau^{(n-1)}\left(\frac{s}{\sqrt{2}}\right) f^{(n-1)}\left(\frac{s}{\sqrt{2}}\right)$$

$$-\frac{1}{2\pi}\int_0^{2\pi} d\gamma \int_0^\infty dt\, \exp[iz_n t - t^2 - 2st(1+2\cos^2\gamma)]$$

$$\times\left\{2\cos^2\gamma(s+t)[\tau^{(n-1)}(+)\, f^{(n-1)}(-) + \tau^{(n-1)}(-)\, f^{(n-1)}(+)]\right.$$

$$-\frac{1}{2}\cos\gamma\left[\left(\frac{s}{t}\right)^{1/2} + \left(\frac{t}{s}\right)^{1/2}\right]$$

$$\times[\tau^{(n-1)'}(+)\, f^{(n-1)}(-) - \tau^{(n-1)}(+)\, f^{(n-1)'}(-)$$

$$\left.+\tau^{(n-1)}(-)\, f^{(n-1)'}(+) - \tau^{(n-1)'}(-)\, f^{(n-1)}(+)]\right\} \tag{3.12}$$

and

$$\delta g^{(n)}(0) = \frac{1}{(\lambda^{(n)})^{1/2}} \int_0^{2\pi} d\gamma \int_0^{\infty} dt\, e^{iz_n t - t^2} \tau^{(n-1)}(t/\sqrt{2})\, f^{(n-1)}(t/\sqrt{2}) \qquad (3.13)$$

To simplify notation for later use, it will be convenient to introduce two bilinear functionals R and Γ, in terms of which the recursions (3.9) and (3.12) can be written as

$$f^{(n)} = R(f^{(n-1)}, f^{(n-1)}) \qquad (3.14)$$

and

$$\tau^{(n)} = \Gamma(f^{(n-1)}, f^{(n-1)}) + \Gamma(f^{(n-1)}, \tau^{(n-1)}) + R(f^{(n-1)}, \tau^{(n-1)}) \qquad (3.15)$$

We see that the recursion equations for the $f^{(n)}$ are autonomous, while those for the $\tau^{(n)}$ also involve the $f^{(n)}$. We will therefore start computing the $f^{(n)}$. Using these results, we can then compute the $\tau^{(n)}$ in essentially the same manner.

Considering the equations (3.9), we realize that the only parameters entering into them are the z_n. Using Eqs. (2.23) and (2.24), those can be expressed as[4]

$$z_n = \frac{(\sqrt{2})^n}{\sqrt{\lambda}} [\zeta - E_{n-1}] \qquad (3.16)$$

This already shows some of the essential features we will exploit. The z_n tend to get extremely large, except when ζ coincides with the eigenvalue E_{n-1} associated with the particular hierarchy n. The $g^{(n)}(0)$ has an explicit factor of $1/(\lambda^{(n)})^{1/2}$. This seems to make it large; however, unless ζ is very close to E_{n-1}, the integrand in (3.10) is rapidly oscillating and the integral is in fact of order $1/z_n$, which exactly offsets the largeness of the prefactor. If, on the other hand, $\zeta = E_{n-1}$, the free Green's function would diverge at this point. For λ nonzero, the e^{-t^2} term ensures that the integral converges and is in fact of order unity. Instead of a pole, we have a large contribution of size $1/(\lambda^{(n)})^{1/2}$, which of course diverges as λ approaches zero.

In this discussion we have ignored the function $f^{(n)}$. The remainder of this paper is essentially devoted to showing that this is justified, i.e., to

[4] The equations in this section so far are valid for $n \leqslant L$ only. Notice, however, that if we *define* z_n through Eq. (3.16), with E_L as given by (2.24), then (3.10) and (3.13) are also valid for $n = L + 1$, so that in the sequel we will not have to treat the last renormalization group step separately from the others. And of course E_L converges to $-1/(2-\alpha)$ at the same rate as E_{L-1}.

showing that the functions $f^{(n)}$ remain for all practical purposes close to one.

From the form of our recursion equations we see that there are three distinct cases that we must distinguish in our discussion:

1. ζ is not near the spectrum of $-\Delta_\alpha$, i.e.,

$$|E_n - \zeta| \geqslant \varepsilon \qquad \text{for all } n$$

Together with (3.16), this guarantees that

$$z_n \geqslant (\sqrt{2})^n \frac{\varepsilon}{\sqrt{\lambda}}$$

2. ζ is near a resonance of finite order, e.g.,

$$|\zeta - E_k| \approx O\left(\frac{\sqrt{\lambda}}{(\sqrt{2})^k}\right)$$

In this case z_{k+1} is small, while all the other z_n are big; in fact,

$$|z_n| \geqslant \frac{(\sqrt{2})^n}{\sqrt{\lambda}} \left(\frac{\alpha}{2}\right)^k$$

3. ζ is close to the accumulation point of the spectrum, i.e.,

$$|\zeta - E_\infty| \approx 0$$

In this case,

$$|z_n| \approx \frac{1}{\sqrt{\lambda}} \left(\frac{\alpha}{\sqrt{2}}\right)^n$$

and is thus growing only for $\alpha > \sqrt{2}$.

We begin by dealing with the easiest case, case 1.

Our first concern is to analyze carefully the function $f^{(1)}(s)$. Since $f^{(0)}(s) = 1$,

$$f^{(1)}(s) = 1 - \frac{1}{2\pi} \int_0^{2\pi} d\gamma \int_0^\infty dt \, \{\exp[iz_1 t - t^2 - 2st(1 + 2\cos^2\gamma)]\} \, 4s\cos^2\gamma \tag{3.17}$$

Evidently, $f^{(1)}(s)$ is an entire function. We will find in the next section global bounds in appropriate domains of the complex s-plane. At this point we already note that as $|s| \to \infty$ with $\Re s > 0$,

$$f^{(1)} \to 1 - \frac{2}{2\pi} \int d\gamma \, \frac{2\cos^2\gamma}{1 + 2\cos^2\gamma} = \frac{1}{\sqrt{5}} \tag{3.18}$$

For z_1 large and s bounded, we can expand $f^{(1)}$ in powers of s/z_1. For the purposes of this section we need the power series expansion of the logarithm of $f^{(1)}$ up to order s^2 for $|s| < c$. Here c will be chosen small compared to z_1. The reason why we expand only to second order is that the coefficients of higher powers of s scale to zero under the leading part of our iteration, i.e., $f(s) \to f(s/\sqrt{2})^2$. If $\ln f(s) = \sum a_k s^k$, then under n iterations of this map it transforms to $\sum a_k [2^n/(\sqrt{2})^{kn}]\, s^k$.

The computation of the expansion proceeds by expanding the $e^{-2st\dots}$ term in the integrand. The rest is straightforward. We get

$$\begin{aligned}\ln f^{(1)}(s) = & -i2\left[\frac{1}{z_1} + O\left(\frac{1}{z_1^2}\right)\right] s - \sqrt{\pi}\, e^{-z_1^2/4} s \\ & - 8\left[\frac{1}{z_1^2} + O\left(\frac{1}{z_1^3}\right)\right] s^2 - i s^2 O(z_1 e^{-z_1^2/4}) \\ & + O\left(\frac{s^3}{z_1^3}\right)\end{aligned} \tag{3.19}$$

We may think of the coefficients of is and s^2 as small perturbative corrections to ζ_1 and λ_1, respectively. Note, however, that (3.19) is qualitatively incorrect for $s > z_1$, where $f^{(1)}$ approaches a constant. The nonperturbatively small coefficients of $-s$ and $-is^2$ have been kept for the sake of completeness. They are of no consequence and will be dropped from now on.

It is now easy to compute the recursion relation for the coefficients of s and s^2 of $\ln f^{(n)}$. We write (for s small)

$$f^{(n)}(s) = \exp[-i\mu_n s - l_n s^2 + O(s^3)] \tag{3.20}$$

and plug this into the equation for $f^{(n+1)}$. Of course, the range of the t-integrations exceeds the domain of validity of (3.20). We will show in the next section that this introduces only errors that are exponentially small in z_n. Given this, we obtain

$$\begin{aligned}\ln f^{(n+1)}(s) = & -i\left(\sqrt{2}\,\mu_n + 2\frac{l_n + 1}{z_{n+1} - \sqrt{2}\,\mu_n}\right) s \\ & - \left[l_n + 8\frac{(1 + l_n)^2}{(z_{n+1} - \sqrt{2}\,\mu_n)^2}\right] s^2 \\ & + O\left(\frac{s}{z_{n+1}^2}, \frac{s^2}{z_{n+1}^3}, \frac{s^3}{z_{n+1}^3}\right)\end{aligned} \tag{3.21}$$

From this we identify μ_{n+1} and l_{n+1}, which to leading order satisfy thus the system of equations

$$\begin{aligned} \mu_{n+1} &= \sqrt{2}\,\mu_n + \frac{2(l_n+1)}{z_{n+1} - \sqrt{2}\,\mu_n} \\ l_{n+1} &= l_n + 8\,\frac{(1+l_n)^2}{(z_{n+1} - \sqrt{2}\,\mu_n)^2} \end{aligned} \tag{3.22}$$

It is convenient to introduce

$$m_n \equiv \frac{1}{(\sqrt{2})^n}\,\mu_n \tag{3.23}$$

and to rewrite (3.22) as

$$\begin{aligned} m_{n+1} &= m_n + \frac{1}{(\sqrt{2})^{n+1}}\,\frac{2(1+l_n)}{(\zeta - E_n)/\sqrt{\lambda} - m_n} \\ l_{n+1} &= l_n + \frac{1}{2^{n+1}}\,\frac{8(1+l_n)^2}{[(\zeta - E_n)/\sqrt{\lambda} - m_n]^2} \end{aligned} \tag{3.24}$$

In the purely perturbative case, i.e., when $|\zeta - E_n| \geqslant \varepsilon$ for all n, this recursion relation converges extremely rapidly due to the presence of the explicit exponentially decaying factors that multiply the nonlinear term. The solution is then virtually equal to the first nonzero term, i.e.,

$$\begin{aligned} m_n &\approx m_1 = \frac{\sqrt{\lambda}}{\zeta - E_0} \\ l_n &\approx l_1 = \frac{2\lambda}{(\zeta - E_0)^2} \end{aligned} \tag{3.25}$$

Thus, except in small regions of approximate size $\sqrt{\lambda}\,(\sqrt{2})^n$ around the eigenvalues E_n of $-\Delta_\alpha$, the Green's function is equal to that of $-\Delta_\alpha$, up to corrections of order $\sqrt{\lambda}$.

We turn now to the more interesting case where $\zeta \approx E_k$. In this case, one can perform the above procedures k times without problem, as before. However, z_{k+1} will be close to zero, and in the computations of f^{k+1} and g^{k+1} we cannot use it as a large parameter.

Let us return to Eq. (3.12) and illustrate what happens in the simplest case, $k=0$. We see that the convergence of the integral is still assured by the e^{-t^2} term, and for s small we may, as before, expand in powers of s.

This time, however, it makes no sense to expand the coefficients in powers of $1/z_1$. We get

$$f^{(1)}(s) = 1 - 2is \int dt \sin(z_1 t)\, e^{-t^2} - s\sqrt{\pi}\, e^{-z_1^2/4} + \tfrac{5}{2} i \sqrt{\pi}\, z_1 e^{-z_1^2/4} s^2 - 10 s^2 \int dt \cos(z_1 t)\, t e^{-t^2} + O(s^3) \tag{3.26}$$

For $z_1 = 0$, this simplifies to

$$f^{(1)}(s) = 1 - \sqrt{\pi}\, s + 5s^2 + O(s^3) \tag{3.27}$$

or,

$$f^{(1)}(s) = \exp\left[-\sqrt{\pi}\, s + \left(5 - \frac{\pi}{2}\right) s^2 + O(s^3) \right] \tag{3.28}$$

Corrections to the coefficients of s and s^2 for small but finite z_1 grow linearly, resp. quadratically, with z_1, and are bounded of order unity for any z_1. Of course this time, the Taylor expansion to second order is good for $s \ll 1$, and therefore the fact that the coefficient of s^2 is positive should not cause us to worry. We will never use this formula for large s. Globally, as we will see in the next section, $f^{(1)}$ remains bounded for positive s.

If z_{k+1} is close to zero instead of z_1, we get qualitatively the same result, this time for $f^{(k+1)}$, since $f^{(k)}$ is equal to one up to perturbative corrections.

After this step, the integrals appearing in all the further iterations have again a rapidly oscillating factor $e^{iz_n t}$. Therefore, all corrections to the leading scaling of $f^{(k+1)}$ from here on will again be perturbative. The leading form of $f^{(n)}$ for small s, and $n > k$, will be of the form

$$f^{(n)}(s) \approx \exp\left[-\sqrt{\pi}\, (\sqrt{2})^{n-k-1}\, s + \left(5 - \frac{\pi}{2}\right) s^2 + O(s^3/(\sqrt{2})^{n-k}) \right] \tag{3.29}$$

Note that the estimates here are not uniform in k. For fixed k we must choose λ sufficiently small, for everything to work. The most subtle question arises when ζ is close to the accumulation point E_∞ of the spectrum of $-\Delta_\alpha$.

A simple consideration can already tell us what to expect. We have seen before that a region of width $\sqrt{\lambda}\, (\sqrt{s})^k$ about the kth eigenvalue requires a nonperturbative estimate in the kth step. On the other hand, the distance between successive eigenvalues shrinks like $(\alpha/2)^k$. Therefore, if $\alpha > \sqrt{2}$, those regions (we would like to think of them as spectral lines) stay well separated as $k \to \infty$, while for $k < \sqrt{2}$ they will eventually overlap, no matter how small l is. In this case, the region near the accumulation point should not be accessible to a perturbative analysis.

How do these features show up in our scheme? We want $\zeta \sim E_\infty$. Up to now we have always ignored the shifts in the position of the resonances due to μ_n, since they were small. Here we can no longer do this. The peaks in the Green's function occur where $z_n - \sqrt{2}\,\mu_{n-1} = 0$, or

$$\zeta = E_{n-1} + \sqrt{\lambda}\, m_{n-1} \tag{3.30}$$

With m_n of order $\sqrt{\lambda}$, the shifts are of order λ. We expect the band edge at an energy

$$\zeta = E_\infty + \sqrt{\lambda}\, e(\lambda) \tag{3.31}$$

where $e(\lambda) \sim \sqrt{\lambda}$ should be tuned so that for this energy $z_n - \sqrt{2}\,\mu_{n-1}$ goes to zero as n goes to infinity.

The recursions for m_n and l_n read in this case

$$\begin{aligned} m_{n+1} &= m_n + \frac{1}{(\sqrt{2})^{n+1}} \frac{2(1+l_n)}{(1/\sqrt{\lambda})(\alpha/2)^{n+1}\,(2-\alpha/2)/(2-\alpha) + e(\lambda) - m_n} \\ l_{n+1} &= l_n + \frac{1}{2^n}\left(\frac{2(1+l_n)}{(1/\sqrt{\lambda})(\alpha/2)^{n+1}\,(2-\alpha/2)/(2-\alpha) + e(\lambda) - m_n}\right)^2 \end{aligned} \tag{3.32}$$

The question is now whether we can choose $e(\lambda)$ consistently. The answer is yes, if $\alpha > \sqrt{2}$, for in this case, assuming first $e(\lambda) - m_n \geqslant 0$, we get

$$m_{n+1} \leqslant m_n + \lambda \left(\frac{\sqrt{2}}{\alpha}\right)^{n+1} c \tag{3.33}$$

implying

$$m_{n+1} \leqslant c\lambda \sum_{1}^{n+1} \left(\frac{\sqrt{2}}{\alpha}\right)^k \tag{3.34}$$

which converges for $\alpha > \sqrt{2}$. Thus, for $e(\lambda)$ bigger than the limit of this series, the assumption is certainly satisfied, and we may choose the true $e(\lambda)$ as the infimum of the set for which this holds. (The l_n have been ignored in this discussion, but it is easy to see that they do not change the picture, since they, too, stay small when m_n does). Note that at this value of the energy ζ,

$$|z_n - \sqrt{2}\,\mu_{n-1}| \geqslant \frac{1}{\sqrt{\lambda}} \left(\frac{\alpha}{\sqrt{2}}\right)^n \frac{2-\alpha/2}{2-\alpha} \tag{3.35}$$

and thus the corresponding $g^{(n)}(0)$ are of order one (see below). For all energies below $E_\infty + e(\lambda)$, $(z_n - \mu_n)$ will cross zero exactly once. At most for one value of n it can happen that $|z_n - \mu_n| \leqslant O(1)$, which would require one nonperturbative step. For all further steps, $(z_n - \mu_n)$ is at least as big as in the case of the band edge, so they can be performed perturbatively.

Thus, for $\alpha > \sqrt{2}$ there is no problem even near the band edge. The case $\alpha = \sqrt{2}$ is marginal and one might analyze it by computing the next order in perturbation theory. The case $\alpha < \sqrt{2}$ should require some non-perturbative analysis and is not within the scope of our present efforts.

Finally, we have to compute the $g^{(n)}(0)$ for the various cases. Whenever z_n is large, this is again done perturbatively; we get

$$g^{(n)}(0) = \frac{1}{(\lambda^{(n)})^{1/2}} \int dt\, e^{iz_n t - t^2} \left[f^{(n-1)}\left(\frac{t}{\sqrt{2}}\right)\right]^2$$
$$= \frac{1}{(\lambda^{(n)})^{1/2}} \frac{1}{z_n - \sqrt{2}\,\mu_{n-1}} + \frac{1}{(\lambda^{(n)})^{1/2}}\, O\left(\frac{1}{(z_n - \sqrt{2}\,\mu_{n-1})^2}\right) \tag{3.36}$$

To leading order, this equals

$$-i \frac{\alpha^n}{2^{n+1}} \frac{1}{\zeta - E_n}$$

which is what one gets for the free Laplacian $-\Delta_\alpha$.

In the case where $z_n \approx 0$, the integral in (3.28) is controlled by the e^{-t^2} term. $f^{(n-1)}$ is still perturbatively close to one at this point, so that the leading contribution is given by

$$g^{(n)}(0) \approx \frac{1}{(\lambda^{(n)})^{1/2}} \int dt\, e^{iz_n t - t^2}$$

The real part of this integral is readily evaluated and gives

$$\Re g^{(n)}(0) \approx \frac{\sqrt{\pi}}{(\lambda^{(n)})^{1/2}} \exp\left[-\frac{2^n}{4\lambda}(\zeta - E_{n-1})^2\right] \tag{3.37}$$

We now have to turn to the computation of the contributions $\delta g^{(n)}(0)$ to the Green's functions and hence to the functions $\tau^{(n)}$. Considering the structure of the recursion relations (3.15), we see that the solution will be of the form

$$\tau^{(n)}(s) = \sum_{i=1}^{n} \tau_i^{(n)}(s) \tag{3.38}$$

where

$$\tau_n^{(n)} = \Gamma(f^{(n-1)}, f^{(n-1)}) \tag{3.39}$$

and, for $i<n$,

$$\tau_i^{(n)}=\Gamma(f^{(n-1)},\tau^{(n-1)})+R(f^{(n-1)},\tau^{(n-1)}) \tag{3.40}$$

Let us first consider $\tau_n^{(n)}$. With $f^{(n-1)}$ given by (3.20), this term reads, on the same level of approximation as we used above,

$$\begin{aligned}\tau_n^{(n)}(s)=&-\frac{1}{2\pi}\int d\gamma\int dt\exp[iz_n t-t^2-2st(1+2\cos\gamma)]\\&\times\exp[-i\sqrt{2}\,\mu_{n-1}(s+t)-l_{n-1}(s^2+t^2+2st(1+2\cos^2\gamma)\\&+O((s,t)^3)]\,4t\cos^2\gamma(1+l_{n-1})+O((s,t)^3)\\=&-[\exp(-i\sqrt{2}\,\mu_{n-1}s-l_{n-1}s^2)]\frac{1}{2\pi}\\&\times\int d\gamma\int dt\exp[i(z_n-\sqrt{2}\,\mu_{n-1})\,t-(1+l_n)\\&\times[t^2-2st(1+2\cos\gamma)]+\cdots]\\&\times 4t\cos^2\gamma(1+l_{n-1})+O((s,t)^3)\end{aligned} \tag{3.41}$$

In the integral in (3.41) we expand again in powers of s. Retaining only the lowest orders in $1/z_n$, this gives

$$\begin{aligned}&\frac{2(1+l_{n-1})}{(z_n-\sqrt{2}\,\mu_{n-1})^2}+i20s\frac{(1+l_{n-1})^2}{(z_n-\sqrt{2}\,\mu_{n-1})^3}\\&\qquad-156s^2\frac{(1+l_{n-1})^3}{(z_n-\sqrt{2}\,\mu_{n-1})^4}+\cdots\\&=\frac{2(1+l_{n-1})}{(z_n-\sqrt{2}\,\mu_{n-1})^2}\exp\left[i\frac{10(1+l_{n-1})}{(z_n-\sqrt{2}\,\mu_{n-1})}s-\frac{28(1+l_{n-1})^2}{(z_n-\sqrt{2}\,\mu_{n-1})^2}s^2+\cdots\right]\end{aligned} \tag{3.42}$$

Thus,

$$\begin{aligned}\tau_n^{(n)}(s)=&\frac{2(1+l_{n-1})}{(z_n-\sqrt{2}\,\mu_{n-1})^2}\\&\times\exp\left[-i\left(\sqrt{2}\,\mu_{n-1}-\frac{10(1+l_{n-1})}{(z_n-\sqrt{2}\,\mu_{n-1})}\right)s\right.\\&\left.-\left(l_{n-1}+\frac{28(1+l_{n-1})^2}{(z_n-\sqrt{2}\,\mu_{n-1})^2}\right)s^2+\cdots\right]\\&\sim\frac{2}{\zeta_n^2}e^{-i\mu_n s-l_n s^2}\end{aligned} \tag{3.43}$$

where in the last line any corrections of order $1/z^n$ are ignored. The total is of order $1/z_n^2$ and therefore a correction of order $\lambda/2^n$ if ζ is not close to E_{n-1}.

The situation is different if $\zeta \approx E_{n-1}$, i.e., we are performing a nonperturbative step. As in the calculation of $f^{(n)}$, we can then not expand in $1/z_n$, but control the integrals by the e^{-s^2} decay. We can expand the result in powers of s to second order with bounded coefficients (and remainder) that are, however, no longer small. For $z_n - \sqrt{2}\,\mu = 0$, the result reads

$$\tau_n^{(n)}(s) = -\frac{1}{1+l_{n-1}} + s\,\frac{\sqrt{\pi}\;5/2}{(1+l_{n-1})^{1/2}} - 13s^2 + O(s^3) \tag{3.44}$$

It is now straightforward to analyze the recursion for the $\tau_i^{(n)}$. Since we know that $\tau_k^{(k)}$ is essentially equal to $c(z_k)\, f^{(k)}$, with $c(z_k) \sim 1/z_k^2$ for z_k large, and bounded otherwise, with $c(0) = -1$, we see that up to minor corrections,

$$\tau_i^{(n)} \approx c(z_i) \prod_{l=i+1}^{n} [1 + c(z_l)]\, f^{(n)} \tag{3.45}$$

The product converges for $n \to \infty$, and unless z_i is small, the whole is a small contribution of order $\lambda/2^i$. Moreover, they are summable over i, so that finally all of $\tau^{(n)}$ is essentially equal to $f^{(n)}$ times a constant that is determined by the contribution with smallest z_i. In particular we get the following results.

1. If $|E_n - \zeta| > \varepsilon^2$ for all n, $|c(z_i)| < \lambda/2^i\varepsilon$ and thus

$$\tau^{(n)} = \sum_{i=1}^{n} \tau_i^{(n)} \sim \frac{\lambda}{\varepsilon^2} f^{(n)} \tag{3.46}$$

and thus

$$\delta g^{(n)}(0) \sim \frac{\lambda}{\varepsilon^2}\, g^{(n)}(0) \tag{3.47}$$

2. If $|\zeta - E_k| = O(\sqrt{\lambda}/(\sqrt{2})^n)$, so that z_{k+1} is not large, the $\tau^{(n)}$ for $n < k+1$ [and thus the $\delta g^{(n)}(0)$ for $n < k+2$] are of the same form as in case 1, i.e., represent corrections of order λ, while for $n \geqslant k+1$,

$$\tau^{(n)} \sim \tau_{k+1}^{(n)} + O(\lambda) \sim O(1)\, f^{(n)} + O(\lambda) \tag{3.48}$$

Thus, for $n \geqslant k+2$, $\delta g^{(n)}(0)$ is of the same order of magnitude as $g^{(n)}(0)$. Of course, this does not concern the dominant singular contribution to the Green's function that in this case is given by $g^{(k+1)}(0)$.

3. For energies near the accumulation point of the free spectrum, the situation is the same as in case 1, if $\sqrt{\alpha} > \sqrt{2}$, since all we needed there was exponential growth of the z_n. Thus, here again the $\delta g^{(n)}(0)$ are a perturbation of order λ compared to the $g^{(n)}(0)$.

Summarizing all those results, we find that, if $\alpha > \sqrt{2}$, to leading order the density of states near an eigenvalue is a Gaussian about this eigenvalue with a rescaled variance, i.e.,[5]

$$\frac{dN_\lambda(E)}{dE} \approx \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\alpha^n} \frac{1}{(\lambda^{(n)})^{1/2}} \exp\left[-\frac{2^n}{4\lambda} (\zeta - E_{n-1})^2 \right] \tag{3.49}$$

away from those regions it is small of order $\sqrt{\lambda}$, and we thus have the formula claimed in Theorem I of section 1. For $\alpha \leqslant \sqrt{2}$ the same formula holds true except in a λ-dependent neighborhood of E_∞ within which our analysis does not give us any control and within which nonperturbative effects will dominate.

4. NONPERTURBATIVE ESTIMATES

In this section we show that the approximations made in the last section are justified, i.e., that all the errors add up only to negligible contributions. We will carry out the explicit analysis for the functions $f^{(n)}$ only. The recursion for the $\tau^{(n)}$ are of essentially the same type and the same analysis applies with minor modifications.

In our perturbative analysis we neglected the contributions from the t-integrals over regions where the argument of $f^{(n)}$ becomes large. This was motivated by the observation that in those regions the $e^{iz_n t}$ oscillates very rapidly, and the resulting terms should be nonperturbatively small. To exploit this in a rigorous manner, we will deform the t-integrals into the complex plane; this will require, however, bounds on the functions $f^{(n)}$ in the corresponding domains of the complex plane. We will start to establish such bounds for $f^{(1)}$, which is given as an explicit integral. Corresponding bounds on $f^{(n)}$ are then proven inductively. Finally, we show the implications of this for $g^{(n)}$.

We split our discussion into the cases $|\zeta - E_n| > \varepsilon$, $\zeta \approx E_k$, and $\zeta \approx E_\infty$.

[5] The Gaussian form of the real part of the $g^{(n)}(0)$ was proven near the corresponding maxima only. In the other regions, this contribution is smaller than the next-order correction. This should be taken into account when reading (3.49).

The function $f^{(1)}(s)$ can be conveniently expressed in terms of the probability integral

$$w(u) = e^{-u^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^u e^{t^2}\,dt\right) \tag{4.1}$$

which has been investigated extensively (see, in particular, ref. 40). We have

$$f^{(1)}(s) = \frac{1}{2\pi}\int d\gamma\,\{1 - \sqrt{\pi}\,2\cos^2\gamma s w[is(1 + 2\cos^2\gamma) + z_1/2]\} \tag{4.2}$$

Using the mean value theorem, we see that we may bound the absolute value of $f^{(1)}(s)$ by a uniform bound on the function

$$1 - \sqrt{\pi}(\rho - 1)\,sw[is\rho + z_1/2] \tag{4.3}$$

for $1 \leqslant \rho \leqslant 3$.

The function $w(u)$ is an entire function, and ref. 40 provides series expansions, asymptotic formulas, and tables for it. Of particular use is the integral representation

$$w(u) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{u - t} \tag{4.4}$$

valid for $\Im u > 0$. The region of $\Im u < 0$ is accessed through the relation

$$w(-u) = 2e^{-u^2} - w(u) \tag{4.5}$$

From this integral representation ref. 40 obtains a quadrature formula for w which is very efficient even for moderately large u:

$$w(u) = \sum_{k=1}^{n}\frac{\lambda_k^{(n)}}{u - x_k^{(n)}} + R_n(u) \tag{4.6}$$

where the $x_k^{(n)}$ are the roots of the nth Hermite polynomial, and $\lambda_k^{(n)}$ are the corresponding coefficients. The convergence of this expansion is extremely rapid, with the error term given by the asymptotic expression

$$R_n(u) = 2(-1)^n\,(\kappa e^{\kappa - 1})^{n + 1/2} \tag{4.7}$$

where

$$\kappa = \frac{[1 - (2n+1)/u^2]^{1/2} - 1}{[1 - (2n+1)/u^2]^{1/2} + 1} \tag{4.8}$$

We will only use this formula with $n=3$. It gives an accuracy of six digits for u of order bigger than five and reads explicitly[40]

$$w(u) \approx \frac{ia_0}{u} + \frac{2ia_3 u}{u^2 - x_3^2} \tag{4.9}$$

where $a_0 = 0.37612639$, $a_3 = 0.09403160$, and $x_3 = 1.2247449$

As a first observation we make the following remark.

Remark 1. Let $s = x + iy$, with $y = kx$. Then, as $x \to +\infty$,

$$f^{(1)}(s) \to 1/\sqrt{5}$$

This result can be read off the asymptotic formula (4.9), or directly from the defining formula. Note that $a_0 + 2a_3 = 1/\sqrt{\pi}$.

We will need to control $f^{(n)}$ on the domain

$$\{s = (\tilde{x} + i\tilde{y})^2 \mid \tilde{x} \in \mathbb{R}, |\tilde{y}| \leqslant \eta\} \tag{4.10}$$

The following bounds on $f^{(1)}$ will be seen later to propagate to the $f^{(n)}$:

Lemma 2. Let z_1 be large; put $s = x + iy$.

(i) If $|s| < \varepsilon z_1$, then

$$|f^{(1)}(s)| \leqslant e^{k_1 y/z_1 - k_2(x^2 - y^2)/z_1^2}$$

where k_1, k_2 are constants of order unity.

(ii) If $|y| < 2v\sqrt{x}$, $x > (2v)^{2/3} |z_1|^{2/3}$, then

$$f^{(1)}(s) \leqslant 1$$

Again, this is easily obtained from the asymptotic expression (4.9).

Corresponding bounds on the functions $f^{(n)}$ will be proven inductively.

Let us now look back at the recursive equations for $f^{(n)}$, (3.9). We will first consider the situation where $|\zeta - E_n| > \varepsilon$ for all n. To make use of the rapid oscillations in the integrals, we deform the integration contours of the t-integrals in the following way: Let $t = (\tilde{t} + i\delta)^2$ with $\tilde{t}$ real and δ a fixed, small real number. The sign of δ is chosen in each step such that the real part of $z_n \delta$ is positive. We first integrate t along the straight line from zero to $8\delta^2 + 6i\delta^2 = (3\delta + i\delta)^2$, and then along the path $t = (\tilde{t} + i\delta)^2$, with $\tilde{t} \geqslant 3\delta$.[6] The second part of the integration path will always produce a non-

[6] This choice of the integration contour is quite arbitrary. All we have to assure is that the small-t part of the contour is sufficiently small so that $[s + t \pm 2(st)^{1/2}]$ with s in the new perturbative region is in the old perturbative region, and that $|\arg t| < \pi/4$. It may be convenient to extend the integration much further along the real axis.

perturbatively small contribution of order $e^{-z_n\delta^2}$. The deformation of the integration contour does not change the integrals, since all functions that appear (for finite n) are entire, and vanish at infinity on the right half-plane.

Suppose we have perturbative control over our function $f^{(n-1)}(s)$ for $|s| < c_{n-1}$, i.e.,

$$f^{(n-1)}(s) = e^{-i\mu_{n-1}s - l_{n-1}s^2 + O(k_{n-1}s^3)}$$

The term $[f^{(n-1)}(s/\sqrt{2})]^2$ can then be evaluated from this formula for $|s| < \sqrt{2}\, c_{n-1}$; the integral over t, ignoring the f's, can be expanded in s/z_n, where $|s| \ll z_n \sim (\sqrt{2})^n/\sqrt{\lambda}$. Finally, the arguments of the $f^{(n-1)}$ remain in the perturbative region during the small-t integrations (i.e., from zero to $8\delta^2 + 6i\delta^2$), provided $|s| < \sqrt{2}\, c_{n-1} - 10\delta^2$. It is therefore consistent to choose c_n as

$$c_n = c(\sqrt{2})^n$$

where c is chosen small compared to $\varepsilon/\sqrt{\lambda}$.

What about the error term? The $[f]^2$ part of the recursion transforms the $O(s^3k_{n-1})$ into $O(s^3k_{n-1}/\sqrt{2})$. The t-integral produces a new error of order $O(s^3/z_n^3)$, and the error terms in the integrals, apart from subleading corrections to the coefficients of s and s^2, contribute to the new error terms of size k_{n-1}/z_n^2, which under our assumption are much smaller (and scale to zero faster) than the leading one. Since $k_1 = k/z_1^3$, it is thus consistent to choose

$$k_n = (\sqrt{\lambda})^3/(\sqrt{2})^n$$

This settles the perturbative estimates for $f^{(n)}(s)$. We still need to show that the integrations over large t indeed contribute only terms of order $e^{-z_n\delta^2}$ to the perturbative computations, and to show that for s large and in the domain of integration, the functions $f^{(n)}$ satisfy the bounds needed in the previous step.

We are now ready to state our main result concerning the functions $f^{(n)}$:

Proposition 1. For all n, $f^{(n)}(s)$ satisfies:

(i) On the domain $D_p(n)$ given by

$$D_p(n) = \{s = (\tilde{x} + i\tilde{y})^2 \mid |s| < c_n, |\tilde{y}| < \eta\} \tag{4.11}$$

with η a suitably chosen (small) constant and $c_n \sim c(\sqrt{2})^n \ll z_n$,

$$f^{(n)}(s) = e^{-i\mu_n s - l_n s^2 + O(s^3 k_n)} \tag{4.12}$$

where $k_n = (\sqrt{\lambda})^3/(\sqrt{2})^n$, and the coefficients μ_n, l_n are to leading order given by the solutions of the recursions (3.18).

(ii) Let $s = x + iy$. For $y < 2\eta\sqrt{x}$ and $x > b_n(2\eta)^{2/3}$, where $b_n \sim (\sqrt{\lambda})^{-2/3}(\sqrt{2})^n$, then

$$|f^{(n)}(s)| < 1 \tag{4.13}$$

Proof. The proof of this proposition is inductive. For $n = 1$, (i) is easily established from Eq. (3.12) using Taylor's formula with remainder. To obtain the $O(s^3/z_1^3)$ estimate for the remainder, one just has to deform the integration contour as indicated above, and exploit the fact that the "large"-t piece of the contour gives only an exponentially small contribution. Part (ii) is obtained using the asymptotic expansion (4.9) and the expression (4.3) for $f^{(1)}$.

For the inductive step we assume (i) and (ii) for $f^{(n-1)}(s)$. The perturbative expression for $f^{(n)}(s)$ for small s then results from the small-t integrals, as discussed above. [Note that we have restricted the regions $D_p(n)$ by the condition $|\tilde{y}| < \eta$. This was mainly done in view of the case $\zeta \approx E_\infty$, as we will see below.] To control the large-t integrals, we must control the $f^{(n-1)}([s + t \pm 2(st)^{1/2}\cos\gamma]/\sqrt{2})$ along the path of integration either by the perturbative estimate (4.12), or by the bound (4.13). The way our regions are chosen, one or the other always apply. The integrals for large t will be estimated in absolute value. This involves the integral

$$\begin{aligned}\int_{\tilde{t} > 3\delta} & dt \exp\{-2z_n\delta\tilde{t} - (\tilde{t}^2 - \delta^2)^2 + 4\delta^2\tilde{t}^2 - 2[x(\tilde{t}^2 - \delta^2) - \delta y\tilde{t}](1 + 2\cos^2\gamma)\} \\ & \times \left\{4|s|\cos^2\gamma\, f^{(n-1)}(+)\, f^{(n-1)}(-) + \left|\left(\frac{s}{t}\right)^{1/2}\right| \, |f^{(n-1)\prime}(+)\, f^{(n-1)}(-) \right. \\ & \left. - f^{(n-1)\prime}(-)\, f^{(n-1)}(+)|\right\}\end{aligned} \tag{4.14}$$

Here the plus signs and minus signs in the arguments of the $f^{(n-1)}$ are shorthand for the $s + t \pm 2(st)^{1/2}\cos\gamma$ of Eq. (3.9). Note that the $f^{(n-1)}$ in the integral are either bounded by one (if t is sufficiently large), or grow at most like $\exp[\mu_n(\tilde{t} + \tilde{x})(\delta + \eta)]$. Equally, the only potentially dangerous factor in front of the f's grows like $\exp(2\delta y\tilde{t})$. Since we have chosen δ and η constant, and $y < \sqrt{x}\,\eta$, with $x \ll z_n$, the decay of the $\exp(-2z_n\delta\tilde{t})$ term dominates all growing terms, and the integral (4.14) gives in fact a contribution of size

$$|s|\, e^{-z_n\delta^2}$$

only.

Estimating $f^{(n)}$ for s large proceeds in a similar manner. We split the integration contour as before. For small t, we expand $f^{(n-1)}$ in a Taylor series with remainder,

$$f^{(n-1)}\left(\frac{t+s\pm 2(st)^{1/2}\cos\gamma}{\sqrt{2}}\right)=f^{(n-1)}\left(\frac{s}{\sqrt{2}}\right) + \frac{t\pm 2(st)^{1/2}\cos\gamma}{\sqrt{2}} f^{(n-1)\prime}\left(\frac{s}{\sqrt{2}}\right) + R^{(2)}(s,t) \tag{4.15}$$

where the second-order remainder is bounded by

$$|R^{(2)}(s,t)| \leqslant \max_{0\leqslant\Theta\leqslant 1}\left|\frac{[t\pm 2(st)^{1/2}\cos\gamma]^2}{4} f^{(n-1)\prime\prime}\left(\frac{s+\Theta(t\pm 2(st)^{1/2}\cos\gamma)}{\sqrt{2}}\right)\right|$$

Thus,

$$f^{(n-1)}(+)\,f^{(n-1)}(-)=[f^{(n-1)}(s/\sqrt{2})]^2+\cdots$$

and

$$f^{(n-1)\prime}(+)\,f^{(n-1)}(-)-f^{(n-1)}(+)\,f^{(n-1)\prime}(-) = 4(st)^{1/2}\cos\gamma[f^{(n-1)\prime}(s/\sqrt{2})]^2+\cdots$$

where the terms we did not write involve extra factors of t and derivatives of the f. Bounds on the derivatives of $f^{(n-1)}$ follow from the Cauchy formulas and the bound (4.13). They imply that

$$f^{(n-1)(k)}(s)\sim\frac{1}{(\sqrt{|s|})^k} \tag{4.16}$$

Since each factor of t produces a factor $1/z_n$, one sees that in fact all those terms produce corrections that are of order $1/z_n$ relative to the leading terms. If we indicate the z dependence of the f by a subscript, this leading form can be written suggestively as

$$f_{z_1}^{(n)}(s)\approx[f_{z_1}^{(n-1)}(s/\sqrt{2})]^2 f_{z_n}^{(1)}(s) \tag{4.17}$$

This will certainly have absolute value less than one in the domain required for $f^{(n)}$, and for ζ_n large enough, the corrections cannot alter this fact.

Finally, we have to show that here, too, the large-t integration produces only an exponentially small correction. The proof proceeds as before. The arguments of $f^{(n-1)}$ now always lie in the regions where the

$f^{(n-1)}$ are bounded by one. The derivatives of $f^{(n-1)}$ satisfy the same (or even better) bounds. Since the imaginary part of s is always smaller than the root of the real part, no growing terms appear in the exponential in (4.14), and we get again, as desired, a bound $e^{-z_n\delta^2}$. This concludes the proof of Proposition 1.

An immediate corollary of Proposition 1 is that away from the spectrum of Δ_α, the perturbative computation of $G^\alpha(\zeta)$ in Section 3 is correct up to exponentially small corrections.

We turn now to the case $\zeta \approx E_k$, for some fixed k. The essential difference compared to the previous situation occurs only in the $(k+1)$th step, where z_{k+1} is near zero, and the integrals cannot be reasonably expanded in inverse powers of it. For $f^{(k)}(s)$ we still have the information of Proposition 1, since all the previous z_n have been large.

The equation for $f^{(k+1)}$ is again (3.9); however, we must control the integral by the e^{-t^2} term rather than by the $e^{iz_{k+1}t}$ term, over which we have no uniform control. Therefore, it is not useful to deform the t-integral into the complex plane, since we get maximal decay in the real direction.

The functions $f^{(k)}$ have perturbative estimates valid in a region $|s| < \varepsilon z_k$. The t-integrals for values of t larger than that will give only terms of order $e^{-\varepsilon^2 z_k^2}$. Expanding as before $f^{(k)}$ about $s/\sqrt{2}$, we get as a leading term

$$f^{(k+1)}(s) \approx [f^{(k)}(s/\sqrt{2})]^2 f^{(1)}_{z_{k+1}\approx 0}(s) \tag{4.18}$$

Again the corrections are irrelevant: For s small (compared to z_k!) the derivatives of $f^{(k)}$ that appear are smaller by factors of at least μ_k than the $f^{(k)}$ themselves. For s large we use that

$$\int dt\, e^{-t^2 - 2st(1+2\cos^2\gamma)} t^n \sim \frac{1}{s^n} \tag{4.19}$$

and that the derivatives of $f^{(k)}$ decay according to (4.16).

To continue, we must thus analyze $f^{(1)}_0(s)$. For s small (compared to one this time), we have the perturbative formula (3.22). For larger values of s, we have the following result.

Lemma 3. For $\Re s \geqslant |\Im s|$,

$$f^{(1)}_0(s)| \leqslant 1 \tag{4.20}$$

Proof. This is in fact a simple consequence of (4.3) and the integral representation (4.4). They allow us to write

$$|f^{(1)}_0(s)| = \left|\frac{2}{\sqrt{\pi}}\int_0^\infty dt\, e^{-t^2}\frac{\rho s^2 + t^2}{\rho^2 s^2 + t^2}\right| \tag{4.21}$$

Since $1 \leqslant \rho \leqslant 3$, the absolute value of this integral is bounded by one, provided only $\Re s^2 \geqslant 0$.

For $f^{(k+1)}$ this leaves us with the following information: We have to distinguish three regions now:

Proposition 2. If $z_{k+1} \approx 0$, $f^{(k+1)}(s)$ satisfies:

(i) For s very small, $s < c \ll 1$, we have the expansion in powers of s,

$$f^{(k+1)}(s) = \exp\left[-\sqrt{\pi}\, s - i\sqrt{2}\,\mu_k s + \left(5 - \frac{\pi}{2}\right) s^2 - l_k s^2 + O(s^3)\right] \tag{4.22}$$

(ii) On the domain $D_p(n)$ [see (4.11)] we do not have complete perturbative control, but since $|f_0^{(1)}(s)| \leqslant 1$ there, we have the bound

$$|f^{(k+1)}(s)| \leqslant \exp(\sqrt{2}\,\mu_k \Im s - \lambda_k \Re s^2) \tag{4.23}$$

which is in fact all we need in the further iterations.

(iii) Finally, with $s = x + iy$, for $y < 2\eta\sqrt{x}$ and $x > \sqrt{2}\, b_k (2\eta)^{2/4}$,

$$|f^{(k+1)}(s)| \leqslant 1$$

Checking through the proof of Proposition 1 with Proposition 2 as input, we see that we can carry these statements into the next hierarchies as before. This implies that the perturbative results of Section 3 are correct also in this case.

Finally, we discuss the band edge. In view of our discussion in Section 3, this means $\zeta = E_\infty + \sqrt{\lambda}\, e(\lambda)$, and perturbatively $|\zeta_n - \mu_n| \geqslant 1/\lambda(\alpha/\sqrt{2})^n c$. For $\alpha > \sqrt{2}$ this is still growing exponentially with n, although not as fast as before. Thus, the size of the perturbative regions $D_p(n)$ is also smaller, but growing, i.e., we will choose the c_n proportional to $(\alpha/\sqrt{2})^n \ll |z_n - \mu_n|$. A problem arises with the large-s bounds on $f^{(n)}$. In Proposition 1 we have the stability bound only for $\Re s > b_n$, where the b_n grow like $(\sqrt{2})^n$. The reason for this is that once such a bound is established for, say, $f^{(1)}(s)$ and $|s| < b$, $[f^{(1)}(s/(\sqrt{2})^n)]^{2^n}$ will satisfy it only for $2/(\sqrt{2})^n < b$. Thus, there appears to be a gap between the perturbative region and the region where we have a stability bound.

However, this problem can be solved as in the previous case by extracting a perturbative bound like (4.23) in the intermediate region, that is, we have the following result.

Proposition 3. For $\zeta \approx E_\infty + \sqrt{\lambda}\, e(\lambda)$, and if $\alpha > \sqrt{2}$, $f^{(n)}$ satisfies:

(i) (4.12) with the coefficients μ_n and l_n computed to leading order from the recursion for $|s| < c_n \sim (\alpha/\sqrt{2})^n$, and $|\tilde{y}| < \eta$.

(ii) A bound

$$|f^{(n)}(s)| \leqslant \exp[(\sqrt{2})^n \mu_1 \Im s - l_1(\Re s^2 - \Im s^2)] \tag{4.24}$$

for s beyond the perturbative region, but $|s| < c(\sqrt{2})^n$.

(iii) $|f^{(n)}(s)| \leqslant 1$ for s as in Proposition 1.

This bound in the intermediate region propagates due to (4.17) and the arguments presented there.

We may summarize our results as follows: In all three cases, the information on $f^{(n)}$ thus obtained suffices to show that the $g^{(n)}(0)$ are given by their perturbative expressions as found in Section 3, plus errors that are exponentially small in z_n. Since the expansion parameters in all cases converge exponentially fast to zero with n, the summations over the hierarchies are convergent and the result is an asymptotic expansion for the Green's function, with bounded error terms. These allow us to continue $G_\infty^\alpha(\zeta)$ as an analytic function of ζ from the upper half-plane to the entire complex plane, for $\lambda \neq 0$ and small enough. This proves Theorem 1.

ACKNOWLEDGMENTS

This work was started at the University of California at Irvine. I would like to thank the Department of Mathematics, and in particular Abel Klein, for the pleasant time I spent there. I am greatly indebted to Fernando Perez, who suggested this particular model, and with whom I had lengthy discussions on this problem, and to Abel Klein for reaching me most of what I know about the Anderson model and for pointing out a number of errors in an earlier draft of this manuscript.

REFERENCES

1. P. W. Anderson, *Phys. Rev.* **109**:1492 (1958).
2. T. Spencer, in *Critical Phenomena, Random Systems, Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam, 1986); D. J. Thouless, in *Critical Phenomena, Random Systems, Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam, 1986).
3. F. J. Dyson, *Commun. Math. Phys.* **12**:91 (1969), **21**:269 (1971).
4. P. Collet and J.-P. Eckmann, *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics* (Lecture Notes in Physics, Vol. 74; Springer, Berlin, 1978).
5. P. M. Bleher and Ja. G. Sinai, *Commun. Math. Phys.* **45**:347 (1975).
6. G. Jona Lasinio, F. Martinelli, and E. Scoppola, *J. Phys. A* **17**:L635 (1984); *Ann. Inst. H. Poincaré A* **42**:73 (1985).
7. R. Livi, A. Maritan, and S. Ruffo, *J. Stat. Phys.* **52**:595 (1988), and references therein.
8. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **88**:151 (1983).

9. J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, *Commun. Math. Phys.* **101**:21 (1985).
10. R. Carmona, A. Klein, and F. Martinelli, *Commun. Math. Phys.* **108**:41 (1987).
11. H. von Dreyfus and A. Klein, *Commun. Math. Phys.* **124**:285 (1989).
12. E. LePage, in *Probability Measures on Groups, VII* (Lecture Notes in Mathematics, No. 1064; Springer, Berlin, 1983).
13. B. Simon and M. Taylor, *Commun. Math. Phys.* **101**:1 (1985).
14. P. March and A. Snitzman, preprint.
15. M. Campanino and A. Klein, *Commun. Math. Phys.* **104**:227 (1986).
16. A. Klein and A. Speis, *J. Funct. Anal.* **88**:211 (1990).
17. F. Constantinescu, J. Fröhlich, and T. Spencer, *J. Stat. Phys.* **34**:571 (1984).
18. S. F. Edwards and D. J. Thouless, *J. Phys. C* **4**:453 (1971).
19. A. Bovier, M. Campanino, A. Klein, and J. F. Perez, *Commun. Math. Phys.* **114**:439 (1988).
20. F. Wegner, *Z. Phys. B* **44**:9 (1981).
21. R. Maier, *J. Stat. Phys.* **488**:425 (1987).
22. M. Kappus and F. Wegner, *Z. Phys. B* **45**:15 (1981).
23. B. Derrida and E. Gardner, *J. Phys.* (Paris) **45**:1283 (1984).
24. A. Bovier and A. Klein, *J. Stat. Phys.* **51**:501 (1988).
25. M. Campanino and A. Klein, UCI preprint (1989).
26. R. Abou-Chacra, P. W. Anderson, and D. J. Thouless, *J. Phys. C* **6**:1734 (1973).
27. H. Kunz and B. Souillard, *J. Phys. Lett.* (Paris) **44**:L411 (1983).
28. M. Benderskii and L. Pastur, *Mat. Sb.* **82**:245 (1970).
29. J. Avron and B. Simon, *Duke Math. J.* **50**:369 (1983).
30. W. Kirsch and F. Martinelli, *J. Phys. A* **5**:2139 (1982).
31. R. Carmona, in *Proceedings of the Ecole d'été de probabilité XIV* (Lecture Notes in Mathematics, No. 1180; Springer, New York, 1986).
32. G. Parisi and N. Sourlas, *J. Phys. Lett.* **41**:2403 (1980).
33. A. J. McKane, *Phys. Lett. A* **76**:22 (1980).
34. A. Klein, L. J. Landau, and J. F. Perez, *Commun. Math. Phys.* **93**:459 (1984).
35. J. M. Luttinger, *J. Math. Phys.* **24**:2070 (1983).
36. K. B. Effetov, *Adv. Phys.* **32**:53 (1983).
37. A. Klein, F. Martinelli, and J. F. Perez, *Commun. Math. Phys.* **106**:623 (1986).
38. K. Gawedzki and A. Kupiainen, in *Critical Phenomena, Random Systems, Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam, 1986).
39. F. A. Berezin, *The Method of Second Quantization* (Academic Press, New York, 1966).
40. V. N. Faddeyeva and N. M. Terent'ev, *Tables of the Probability Integral for Complex Argument* (Pergamon Press, New York, 1961).